Astronomy 45. Introduction to Astrophysics

Problem set 5 Due Monday, May 1

1. Two-body orbit equations

a) Show by explicit calculation that the kinetic energy

$$KE \equiv \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$$

is equal to $E + GM\mu/r$. (Hint: use the orbital equations for $\dot{\theta}$ and $\dot{r}$.) Explain why this result it consistent with E being, in fact, the total energy.

b) Evaluate

$$\int_0^{2\pi} r d\theta$$

around a closed elliptical orbit, and express the answer in terms of r_0 and ϵ. (Hint: you will encounter a trigonemetric integral probably requiring you to refer to a table of integrals.)

2. Virial Theorem applied to an elliptical orbit

a) Show that the time average of the potential energy over one orbital period τ,

$$\langle V \rangle = \frac{1}{\tau}\int_0^{\tau}\left(-\frac{GM\mu}{r}\right) dt$$

can be reduced to the integral of problem 1b (with some other stuff in front). Hint: use the equation for $\dot{\theta}$ to change variables in the integral.

b) Using the result from 1b, evaluate $\langle V \rangle$ and show that it equals $\langle 2E \rangle$. Hint: You'll need several of the equations relating $E, L, a, \epsilon, \tau, r_0$, etc. for 2-body orbits; see notes. (This is not easy.)

c) (Do this even if you've missed the previous parts!) If $\langle V \rangle = \langle 2E \rangle$, as shown in 2b, show that the Virial Theorem is satisfied.

3. Lane Emden Equation

a) Solve analytically the Lane-Emden equation in the special case $n = 0$. (Be sure to impose proper boundary conditions at $\xi = 0$.)

b) Verify by substitution that for $n = 5$ the formula

$$\phi = \frac{1}{\left(1 + \frac{1}{3}\xi^2\right)^{1/2}}$$

satisfies the Lane Emden equation and has the correct boundary conditions.