Astronomy 45. Introduction to Astrophysics

Problem set 4 Due Friday, April 4

1. Virial Theorem for Elliptical Orbits

a) Show that the time average of the potential energy over one orbital period τ can be reduced to the form

$$\langle V \rangle = \frac{1}{\tau}\int_0^\tau \left(\frac{GM\mu}{r}\right) dt = A\int_0^{2\pi} r d\theta,$$

where A is a constant factor.

b) Evaluate the integral from part (a), using the fact that

$$\int_0^{2\pi} \frac{dx}{1 + a\cos x} = \frac{2\pi}{\sqrt{1-a^2}}.$$

Show that the resulting expression is equal to $\langle 2E \rangle$, where E is the total orbital energy.

c) If $\langle V \rangle = \langle 2E \rangle$, show that the virial theorem is satisfied.

2. Disruption by Tides

a) Find the separation d between two small bodies, each of mass m, lined up with a large body of mass M and a distance R from the nearest of the small bodies, such that the graviational attraction between the small bodies is just equal to the differential graviational force between them caused by their attraction to the larger body. Ignore any relative motions.

b) Imagine now that a small body of mass m and radius r is falling radially ($L = 0$) towards a large body of mass M. Approximately at what distance between them, the smaller mass will start to be torn apart by tidal forces? (Hint: this problem is very closely related to part (a).)

c) Based on your result for part (b), could any of the solar system planets come close enough to the Sun to be tidally disrupted? What if the Sun had the same mass, but 100 times smaller radius (typical white dwarf star)? The planetary radii and masses range from $0.0017R_\odot$ and $6.6 \times 10^{-9}M_\odot$ (Pluto) to $0.10R_\odot$ and $9.5 \times 10^{-4}M_\odot$ (Jupiter).

d) One of the possible explanations for quasars (very far away and very bright objects) involves a giant black hole in the center of a galaxy much like our Milky Way, tidally disrupting stars that happen to pass near by. What is the *minimum* mass black hole can have to disrupt a Sun-like star?

We can observe this disruption only if it happens outside the black hole horizon (radius inside which even light cannot escape the hole's gravity), which is given by the expression

$$R_h = \frac{2GM_{BH}}{c^2},$$

where c is the speed of light. Calculate the *maximum* mass a black hole can have for us to observe the disruption of stars like our Sun. Are larger mass black holes better or poorer at disrupting stars?

3. Energy Lost During Accretion

Suppose that two stars are bound together in a binary system, and that one of these stars (the *primary*) is losing mass to the other (the *secondary*) at a rate $\dot{m}$. Take the radius of the secondary star to be R_2 and its mass to be M_2.

a) Show that the rate of heating of the secondary because of conversion of the kinetic energy of the infalling matter to heat is

$$L = \alpha \frac{GM_2 \dot{m}}{R_2},$$

where $\alpha = 1$ if the infalling matter is assumed to be falling in from rest at infinite distance. In a real binary system, what processes occur to make $\alpha < 1$?

b) In an equilibrium situation this energy is then lost through radiation. Assuming that the emerging radiation is a blackbody, show that the surface temperature of the secondary star is

$$T_2 = \left(\frac{\alpha G M_2 \dot{m}}{4\pi\sigma R_2^3} \right)^{1/4}.$$

Assume that the accreting material is distributed uniformly over the surface of the secondary and the internal thermonuclear energy production of the secondary is negligible.

c) If the secondary star resembles our Sun (i.e. has mass $M_\odot$ and radius $R_\odot$), and is accreting material at a rate $\dot{m} = 10^{-8} M_\odot/\text{yr}$, what would be the observed luminosity and characteristic frequency? Compare the accretion luminosity and the ordinary solar luminosity; is the rate of energy input from accretion significant compared with the internal energy generation rate of such a star?

d) More interesting cases arise when the primary is a white dwarf (of mass $\sim 1 M_\odot$ and radius $\sim 10^9$ cm) or a neutron star (of mass $\sim 1.4 M_\odot$ and radius $\sim 10^6$ cm). In what wavebands (radio, IR, optical, UV, X-ray, or γ-ray) should observations be made to find such systems, if the accretion rate is the same as in part (c)? How luminous (in units of $L_\odot$) are these systems?