Astronomy 45. Introduction to Astrophysics

Problem 1

a) Let ω and $\omega_\oplus$ be the orbital angular velocities of a planet and of Earth respectively. Then the angular velocity of the planet in the frame of reference in which Earth is stationary is simply

$$\omega' = |\omega - \omega_\oplus| \,.$$

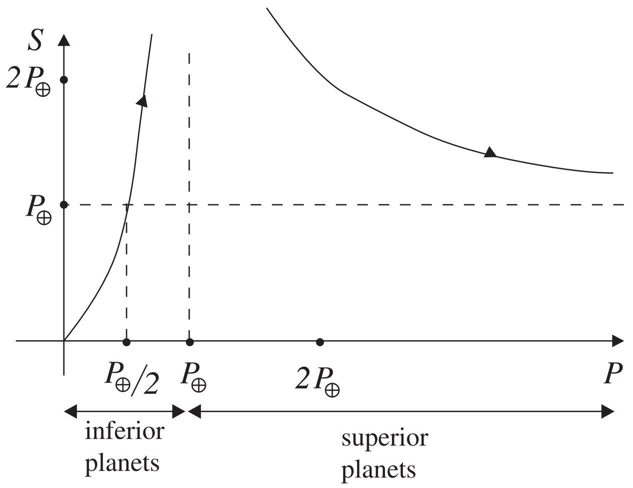
By definition of the synodic period, $S = \frac{2\pi}{\omega'}$, and of course $P = \frac{2\pi}{\omega}$ and $P_\oplus = \frac{2\pi}{\omega_\oplus}$. Then we get

$$\frac{1}{S} = \left|\frac{1}{P} - \frac{1}{P_\oplus}\right| \,.$$

For *inferior* planets, $P < P_\oplus$, so $\boxed{\frac{1}{S} = \frac{1}{P} - \frac{1}{P_\oplus}}$.

For *superior* planets, $P > P_\oplus$, so $\boxed{\frac{1}{S} = \frac{1}{P_\oplus} - \frac{1}{P}}$.

b) One way to solve this problem is of course to compute the synodic period for every planet. However, one can do it in a more general way. From part (a) we know that $S = \frac{P_\oplus P}{|P - P_\oplus|}$. We can plot this relationship to see how S changes with P.



For $P \gg P_\oplus$, $S \to P_\oplus$. For $P \ll P_\oplus$, $S \to P$.

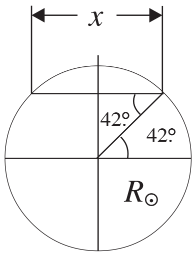
From this it is clear, that interior planets with sidereal periods less than $\frac{P_\oplus}{2}$ have the shortest synodic periods. Only Mercury satisfies this criterion, so its synodic period is the shortest

$$\boxed{S_M = 0.317 \text{ year} = 115.9 \text{ days}}$$

Note that Venus, the next planet after Mercury has $P = 0.62\,\text{yr}$ so if Mercury did not exist, computing the synodic period for every planet would be the only way to solve this problem.

Problem 2

a) Let's do this problem in terms of angles and angular velocities.



Angular velocity of Venus in Earth's reference frame is

$$\omega_V' = \omega_V - \omega_\oplus = \underbrace{\frac{2\pi}{P_V}}_{\substack{\text{sidereal period}\\ \text{of Venus}}} - \underbrace{\frac{2\pi}{P_\oplus}}_{\substack{\text{sidereal period}\\ \text{of Earth}}} .$$

We know that

$$\begin{aligned} P_V = 0.615\,\text{yr} &= 0.615\,\text{yr} \times 365.26\,\text{days/yr} \times 24\,\text{hours/day} \\ &= 53913 \text{ hours} \end{aligned}$$

and

$$\begin{aligned} P_\oplus &= 1\,\text{yr} \times 365.26\,\text{days/yr} \times 24\,\text{hours/day} \\ \Rightarrow \omega_V' &= 4.49 \times 10^{-4}\,\text{rad/hour} . \end{aligned}$$

Now, the angular distance Venus has to travel is just

$$x = 2R_\odot \cos 42^\circ = D_\odot \cos 42^\circ = 32' \cos 42^\circ = 23'78$$

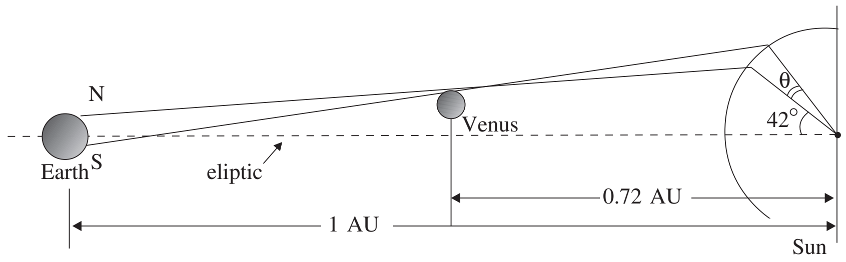
or

$$x = 6.92 \times 10^{-3}\,\text{rad} .$$

Therefore, the crossing time is

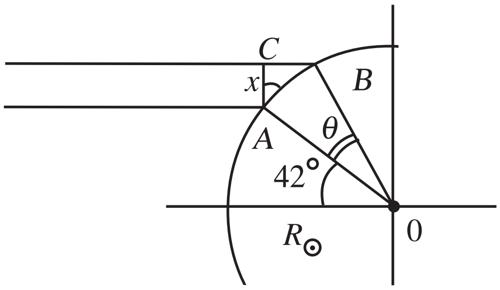
$$\Delta t = \frac{x}{\omega_V'} = \frac{6.92 \times 10^{-3}\,\text{rad}}{4.49 \times 10^{-4}\,\text{rad/hour}} = \boxed{15.4 \text{ hours}}$$

b) This is a very trigonometry-intensive problem. To get a general picture of what is happening, consider the following figure: (the figure is of course not to scale)



So the southern observer will actually see Venus crossing more to the North of the Sun, at some latitude $(42° + \theta)$. And our object is to find θ.

Now, the radius of the Sun is very small compared with 1 AU ($\frac{R_\oplus}{1\,\mathrm{AU}} = 0.0047 < 1\%$), so we can consider that the line from Earth to Venus to the Sun is actually parallel to the ecliptic.



The distance x is easily computed:

$$\frac{8000\,\mathrm{km}}{(1-0.72)\,\mathrm{AU}} = \frac{x}{0.72\,\mathrm{AU}} \quad \Rightarrow \quad x = 20571\,\mathrm{km}$$

Now we assume that θ is very small, so that the segment AB is roughly perpendicular to AO, and the angle BAC is 42°. Then $AB = \frac{x}{\cos 42°} = 27682\,\mathrm{km}$. Again, since θ is small, the arclength between the points A and B is roughly equal to AB, so

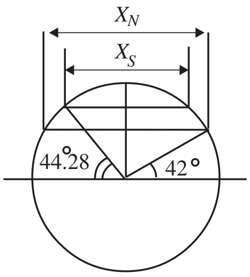
$$\theta \simeq \frac{AB}{R_\odot} = \frac{27682\,\mathrm{km}}{6.96\times 10^{10}\,\mathrm{cm}} = \frac{2.77\times 10^{9}\,\mathrm{cm}}{6.96\times 10^{10}\,\mathrm{cm}} = 0.0398\,\mathrm{rad} = \boxed{2°.28}$$

So θ is indeed small and our approximations are valid. Thus, the S observer sees Venus cross at $\boxed{44.28° N.}$

c) We assume that both observers see Venus cross the mid line of the Sun at the same time. Then the difference in the time when Venus touches the Sun's disk is just

$$\Delta t = \frac{x_N - x_S}{2\omega'_V} = \frac{2R_\odot \cos 42° - 2R_\odot \cos 44°.28}{2\omega'_V} = \frac{R_\odot}{\omega'_V}(\cos 42° - \cos 44°.28)$$

$$= \frac{16'}{4.49\times 10^{-4}\,\mathrm{rad/hour}} \times 0.0272 = 0.282\,\mathrm{hr} = 16\,\mathrm{min}\,55.32\,\mathrm{sec}\ .$$



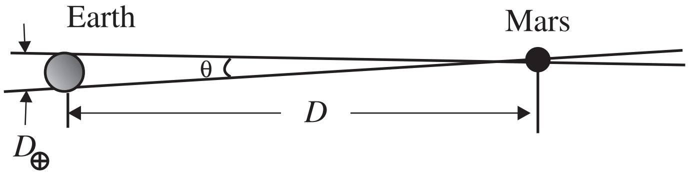
Thus, the S observer sees Venus touch the sun at $\boxed{11{:}16{:}55.32}$.

Problem 3

a)

The parallax angle of Mars is $\theta = 33''.6 = 1.63 \times 10^{-4}$ rad. Since θ is small,

$$D \simeq \frac{D_\oplus}{\theta} = \frac{2 \times 6.38 \times 10^8 \text{ cm}}{\theta} = \boxed{7.83 \times 10^{12} \text{ cm}} = 0.52 \text{ AU} .$$



b) Let's first show that Earth's rotation can be ignored. Orbital velocities of Earth and Mars are

$$\begin{aligned} v_\oplus &= \omega_\oplus a_\oplus , \quad & a_\oplus &= \text{distance from Earth to the Sun} \\ v_M &= \omega_M a_M , \quad & a_M &= \text{distance from Mars to the Sun} \end{aligned}$$

But

$$\begin{aligned} a_\oplus &= 1 \text{ AU} = 1.496 \times 10^{13} \text{ cm} \\ a_M &= a_\oplus + D = 1.496 \times 10^{13} \text{ cm} + 7.83 \times 10^{12} \text{ cm} = 2.28 \times 10^{13} \text{ cm} \end{aligned}$$

and

$$\begin{aligned} \omega_\oplus &= \frac{2\pi}{P_\oplus} = 1.99 \times 10^{-7} \text{ rad/s} \\ \omega_M &= \frac{2\pi}{P_M} = \frac{2\pi}{1.8809 \text{ yr}} = 1.06 \times 10^{-7} \text{ rad/s} \end{aligned}$$

so

$$\begin{aligned} v_\oplus &= 2.977 \times 10^6 \text{ cm/s} = 29.77 \text{ km/s} \\ v_M &= 2.41 \times 10^6 \text{ cm/s} = 24.13 \text{ km/s} . \end{aligned}$$

The relative velocity between Earth and Mars-to-Sun line (we are assuming that mars is fixed and Earth is flying past it) at the time of opposition is

$$v_R = v_\oplus - v_M = 5.64\,\mathrm{km/s}.$$

The velocity of a point on Earth's equator due to Earth rotation is

$$v = R_\oplus \frac{2\pi}{1\mathrm{day}} = 6.38 \times 10^8\,\mathrm{cm} \times 7.27 \times 10^{-5}\,\mathrm{rad/s} = 0.46\,\mathrm{km/s},$$

where $R_\oplus$ is the Earth's radius. So v is ~ 12 times smaller than v_R and since we are looking for a 10% accuracy, it is negligible.

Now let's comput how closely the clocks must be synchronized. The astronomers in 1672 would assume that they know the baseline ($D_\oplus$) well. So to measure D to within 10%, we must measure θ to within 10%. Since we know what the distance to Mars really is, we can compute what that means for the time synchronization. $\theta = \frac{\mathrm{Baseline}}{D}$, where Baseline $= D_\oplus \pm v_R \Delta t$, where Δt is the difference between the observer's clocks. So

$$\Delta\theta = \frac{v_R \Delta t}{D} < 0.1\theta, \quad \text{and} \quad \Delta t < \frac{0.1\theta D}{v_R}.$$

Assuming that values in part (a) for θ and D are correct, we get

$$\Delta t < \frac{0.1 \times 1.63 \times 10^{-4} \times 7.83 \times 10^7\,\mathrm{km}}{5.64\,\mathrm{km/s}} = 226\,\mathrm{sec} = \boxed{3.8\,\mathrm{min}}$$

So the observers' clocks had to by synchronized to $\sim 4\,\mathrm{min}$!

Problem 4

a)

	α	δ
α Centauri	$219^\circ.900$	$-60^\circ.83$
Prox. Centauri	$216^\circ.575$	$-62^\circ.467$

(We converted α to degrees.)

Remember that the coordinates on a unit sphere are:

$$\begin{aligned} z &= \sin\delta \\ x &= \cos\delta\cos\alpha \\ y &= \cos\delta\sin\alpha \end{aligned}$$

Let θ be the angle between the two stars. Then

$$\tan\left(\frac{\theta}{2}\right) = \left|\frac{\hat{r}_1 - \hat{r}_2}{\hat{r}_1 + \hat{r}_2}\right|,$$

where $\hat{r}_1$ and $\hat{r}_2$ are the position vectors for the two stars. Then

$$\tan\left(\frac{\theta}{2}\right) = \left[\frac{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}{(x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2}\right]^{1/2} =$$

$$\tan\left(\tfrac{\theta}{2}\right) = \left[\tfrac{(\cos\delta_1\cos\alpha_1 - \cos\delta_2\cos\alpha_2)^2 + (\cos\delta_1\sin\alpha_1 - \cos\delta_2\sin\alpha_2)^2 + (\sin\delta_1 - \sin\delta_2)^2}{(\cos\delta_1\cos\alpha_1 + \cos\delta_2\cos\alpha_2)^2 + (\cos\delta_1\sin\alpha_1 + \cos\delta_2\sin\alpha_2)^2 + (\sin\delta_1 + \sin\delta_1)^2}\right]^{1/2} .$$

Expanding the squares and using basic trigonometry laws ($\cos^2 x + \sin^2 x = 1$; $\cos x \cos y + \sin x \sin y = \cos[x - y]$), we get

$$\boxed{\tan\left(\tfrac{\theta}{2}\right) = \left[\tfrac{1 - \sin\delta_1\sin\delta_2 - \cos\delta_1\cos\delta_2\cos(\alpha_1 - \alpha_2)}{1 + \sin\delta_1\sin\delta_2 + \cos\delta_1\cos\delta_2\cos(\alpha_1 - \alpha_2)}\right]^{1/2}}$$

Substituting the values for δ_1, δ_2 and α_1, α_2, we finally obtain

$$\tan\left(\frac{\theta}{2}\right) = 0.0198$$

and

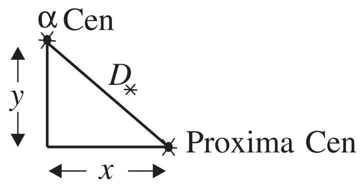
$$\theta = 2\arctan(0.0198) = \boxed{2^\circ.27 = 0.04\,\mathrm{rad.}}$$

b) Projected distance (distance in the plane of Proxima Centauri) is

$$\begin{aligned} x &= \theta \cdot D_{\mathrm{Pr.\ Cen}} = 0.04 \times 4.3\,\mathrm{lyr} = \boxed{0.17\,\mathrm{lyr}} \\ y &= 4.4\,\mathrm{lyr} - 4.3\,\mathrm{lyr} = 0.1\,\mathrm{lyr.} \end{aligned}$$

Then

$$D_* = \sqrt{x^2 + y^2} = \boxed{0.2\,\mathrm{lyr}}$$



c) By definition of the apparent magnitude:

$$m_{\mathrm{tot}} = \mathrm{const} - 2.5\log F_{\mathrm{tot}}$$

where

$$F_{\mathrm{Tot}} = F_1 + F_2 + F_3 = \text{sum of the fluxes}$$

But

$$\begin{aligned} m_1 &= \mathrm{const} - 2.5\log F_1; \Rightarrow F_1 = 10^{(\mathrm{const} - m_1)}/2.5 \\ m_2 &= \mathrm{const} - 2.5\log F_2; \Rightarrow F_2 = 10^{(\mathrm{const} - m_2)}/2.5 \\ m_3 &= \mathrm{const} - 2.5\log F_3; \Rightarrow F_3 = 10^{(\mathrm{const} - m_3)}/2.5\ . \end{aligned}$$

So

$$
\begin{aligned}
m_{\text{Tot}} &= \text{const} - 2.5\log\left[10^{\text{const}/2.5}\left(10^{-m_1/2.5} + 10^{-m_2/2.5} + 10^{-m_3/2.5}\right)\right] \\
&= \cancel{\text{const}} - 2.5\;\frac{\cancel{\text{const}}}{\cancel{2.5}} - 2.5\log\left[10^{-m_1/2.5} + 10^{-m_2/2.5} + 10^{-m_3/2.5}\right] \\
&= -2.5\log\left[10^{-m_2/2.5} + 10^{-m_2/2.5} + 10^{-m_3/2.5}\right]\ .
\end{aligned}
$$

We know that $m_1 = 0$, $m_2 = 1.3$, and $m_3 = 11.0$ Then $\boxed{m_{\text{Tot}} = -0.287}$.

The total flux in the visual band can be computed using formula on p. 17 in the notes.

$$F_C = 10^{-4.42} \times 0.098\mu \times 10^{-0.4(-0.287)} \frac{\text{erg}}{\text{cm}^2\,\text{s}\,\mu} = \boxed{4.4 \times 10^{-6}\,\frac{\text{erg}}{\text{cm}^2\,\text{s}}}$$

d) $M = m - 5\log\left(\frac{D}{10\,\text{pc}}\right)$

α **Cen**:	$m = 0.0, D = 4.4\,\text{lyr} = 1.35\,\text{pc}$	$\Rightarrow$	$\boxed{M = 4.35}$
Proxima Cen:	$m = 11.0, D = 4.3\,\text{lyr} = 1.32\,\text{pc}$	$\Rightarrow$	$\boxed{M = 15.4}$
Third star:	$m = 1.3, D \simeq 4.35\,\text{lyr} = 1.33\,\text{pc}$	$\Rightarrow$	$\boxed{M = 5.67}$

To get the spectral types and masses of these stars, we use the table on page 18 of the notes.

α Cen has spectral type between F and G, and mass $\sim 1.2M_\odot$.

Proxima Cen has spectral type M and mass $\sim 0.2M_\odot$.

The third companion has spectral type between G and K and mass $\sim 0.9M_\odot$.

This part might not make too much sense to you now, but later you will learn more about it.