Astronomy 45. Introduction to Astrophysics

Problem 1

a) In a narrow wavelength band, the number of photons is simply proportional to flux: $F \propto N$. When we detect N photons from a star, the accuracy of our measurement is roughly $\pm\sqrt{N}$. But how to go from number of photons to magnitudes?

From the definition of an apparent magnitude

$$m = -2.5 \log F + const.$$

Then

$$m + 0.02 = -2.5 \log (F/1.019) + const,$$

so to measure a magnitude to an accuracy of 0.02, we need to measure F to within a factor of 1.019. Thus we have

$$F \propto N \quad \text{and} \quad 1.019F \propto N + \sqrt{N}$$

Dividing the two equations we get

$$1.019 = \frac{N + \sqrt{N}}{N},$$

with the result $\boxed{N \simeq 2800}$.

b) Total energy received at the telescope can be written

$$E = h\nu_B N = F_B A \Delta t,$$

where A is area of the telescope, F_B is the flux and ν_B is the average frequency in the B band. We can find F_B using the equation on page 17:

$$F_B = 10^{-4.18} 0.098\mu\, 10^{-0.420} = 6.47 \times 10^{-14}\,\text{ergs/cm}^2\text{s}.$$

Area of the telescope is $A = \pi(50\,\text{cm})^2 = 7.85 \times 10^3\,\text{cm}^2$, and $\nu_B = c/\lambda_B = 3 \times 10^{10}\,\text{cm/s}/4.4 \times 10^{-5}\,\text{cm} = 6.82 \times 10^{14}\,\text{Hz}$. Then time required is just

$$\Delta t = \frac{h\nu_B N}{F_B A} = \frac{6.63 \times 10^{-27}\,\text{ergs s} \times 6.82 \times 10^{14}\,\text{Hz} \times 2800}{6.47 \times 10^{-14}\,\text{ergs/cm}^2\text{s} \times 7.85 \times 10^3\,\text{cm}^2} \simeq \boxed{25\,\text{sec}}.$$

c) By definition of brightness, $I_\lambda \simeq F_B/(\Delta\lambda\,\Omega)$, where $\Delta\lambda$ is the bandwidth of the B band. Again using the equation on page 17 of the notes we get

$$F_B = 10^{-4.18} 0.098\mu\, 10^{-0.422.5} = 6.47 \times 10^{-15}\,\text{ergs/cm}^2\text{s}.$$

The flux is spread out over 1 arcsecond which is 2.35×10^{-11} ster, and the bandwidth is $980\mathring{A}$. Then

$$I_\lambda = \frac{6.47 \times 10^{-15}\,\mathrm{ergs/cm^2 s}}{980\mathring{A} \times 2.35 \times 10^{-11}\,\mathrm{ster}} = \boxed{2.81 \times 10^{-7}\,\mathrm{ergs/(cm^2\,ster\,s\,\mathring{A})}}.$$

d) This part is very similar to part (a), the only difference is that we need to include statistical fluctuations of the total number of photons received, the sky photons as well as the star photons (we can always subtract the *average* sky background). To do that, we need to know the ratio of the sky and star fluxes:

$$\frac{F_{sky}}{F_\star} = \frac{10^{-0.422.5}}{10^{-0.424}} \simeq 4.$$

Then, the ratio of the number of sky photons, N_{sky}, to the number of star photons, $N_\star$ is also 4.

From part (a) we know that accuracy of 0.02 madnitudes requires the knowledge of star flux to a factor of 1.019. Then we have

$$1.019 = \frac{N_\star + \sqrt{N_\star + N_{sky}}}{N_\star}.$$

But we know that $N_{sky}/N_\star = 4$, so the equation above becomes trivial to solve, and gives us $\boxed{N_\star \simeq 14000}$. We need a lot more photons, as could be expected.

e) This part is identical to part (b), with slightly different numbers. We get

$$F_B = 10^{-4.18} 0.098\mu\, 10^{-0.424} = 1.63 \times 10^{-15}\,\mathrm{ergs/cm^2 s},$$

$$A = \pi (200\,\mathrm{cm})^2 = 1.26 \times 10^5\,\mathrm{cm}^2,$$

and

$$\Delta t = \frac{h\nu_B N}{F_B A} = \frac{6.63 \times 10^{-27}\,\mathrm{ergs\,s} \times 6.82 \times 10^{14}\,\mathrm{Hz} \times 14000}{1.63 \times 10^{-15}\,\mathrm{ergs/cm^2 s} \times 1.26 \times 10^5\,\mathrm{cm}^2} \simeq \boxed{307\,\mathrm{sec}}.$$

Problem 2

a) The force, f, on the sail is equal to the rate of momentum transfer, p, from the photons to the sail

$$f = M_s a = \frac{dp}{dt},$$

where M_s is the sail mass, and a is the resulting acceleration.

1. *Black Sail*: the incident photons are absorbed.
   Since all the photons are traveling in one direction perpendicular to the sail, the momentum transfer from each photon is $\Delta p = h\nu/c$, where ν is the photon frequency. The total rate of momentum transfer is then

   $$\frac{dp}{dt} = \frac{\sum_A \Delta p}{\Delta t} = \frac{\sum_A h\nu/c}{\Delta t},$$

   where that sum is computed over all photons incident on the sail of area A in time interval Δt. It is easy to see that the right hand side of this equation is simply FA/c, where

   $$F = \frac{\sigma T_\odot^4\, 4\pi R_\odot^2}{4\pi D_\oplus^2} = \frac{\sigma T_\odot^4\, R_\odot^2}{D_\oplus^2}$$

   is the Solar flux at $D_\oplus = 1$ AU.

   The mass of the sail can be written as $M_s = \rho A w$, where ρ is the density and w is the thickness. Then our first equation becomes

   $$\rho A w\, a = \frac{A\sigma T_\odot^4\, R_\odot^2}{cD_\oplus^2}.$$

   The area of the sail cancels out and the final expression for acceleration is

   $$a = \frac{\sigma T^4 R_\odot^2}{D_\oplus^2 c\rho w};$$

   $$a = \frac{(5.67\times10^{-5}\,\frac{\mathrm{ergs}}{\mathrm{s\,cm^2K^4}})\,(5700\,\mathrm{K})^4\,(7\times10^{10}\,\mathrm{cm})^2}{(1.5\times10^{13}\,\mathrm{cm})^2\,(3\times10^{10}\,\frac{\mathrm{cm}}{\mathrm{s}})\,(2\,\frac{\mathrm{g}}{\mathrm{cm^3}})\,(10^{-3}\,\mathrm{cm})} = \boxed{0.024\,\mathrm{cm/\,s^2}}.$$

2. *Shiny Sail*: the incident photons bounce off.
   For the shiny sail we get twice the momentum transfer compared to the black sail, so our acceleration is doubled as well:

   $$\boxed{a = 0.048\,\mathrm{cm/\,s^2}}.$$

b) Force per unit area stays the same, since the solar flux does not change. So if the acceleration is decreased by a factor of 2, that means the total mass is increased by a factor of 2:

$$M_s + M_{\mathrm{craft}} = 2M_s.$$

Then $M_{\mathrm{craft}} = M_s$ and since the area of the sail is πR^2, where R is the radius, we have

$$R = \sqrt{\frac{M_{\mathrm{craft}}}{\rho\pi w}} = \boxed{138\ \ \mathrm{m}}.$$

Note that it does not matter whether the sail is black or shiny in this case.

Problem 3

a) Angular resolution is defined as

$$\theta = \frac{\lambda}{D}\,.$$

For visual light $\lambda \sim 5000\mathring{A}$, so

$$D = \frac{\lambda}{\theta} = \frac{5000\mathring{A}}{1''} = \frac{5 \times 10^{-5}\,\mathrm{cm}}{4.85 \times 10^{-6}\,\mathrm{rad}} \simeq \boxed{10\ \mathrm{cm}}.$$

This is a very *small* telescope. The reason we need much larger telescopes is not to get better resolution, but to collect more photons in less time — very important for faint sources.

b) Using the relations $f/\# = L/D$ and $\delta\theta = \delta l/L$ (see notes for the definition of symbols), we get

$$\delta\theta = \frac{\delta l}{L} = \frac{\delta l}{D \cdot f/\#}$$

1. δl = size of a pixel $= 1.5 \times 10^{-5}$ m. Since $D = 5\,\mathrm{m}$ and $f/\# = 1.4$ we get

   $$\delta\theta_p = \frac{1.5 \times 10^{-5}\,\mathrm{m}}{5\,\mathrm{m} \cdot 1.4} = 2.14 = 10^{-6}\,\mathrm{rad}.$$

   So one pixel corresponds to $\sim (\delta\theta_p)^2 = \boxed{4.6 \times 10^{-12}\,\mathrm{sterradian}}$.

2. δl = size of the whole array $= 1.5 \times 10^{-5}\,\mathrm{m} \times 800$, so

   $$\delta\theta_{FOW} = 2.14 \times 10^{-6}\,\mathrm{rad} \times 800 = 0.0017\,\mathrm{rad}$$

   and the total field of view is $\sim \boxed{2.9 \times 10^{-6}\,\mathrm{sterradian}}$.

c) Diffraction limited resolution angle is

$$\delta\theta = \frac{\lambda}{D}.$$

But because of the finite pixel size, δl, we cannot resolve angles less than

$$\delta\theta = \frac{\delta l}{L} = \frac{\delta l}{D \cdot f/\#}.$$

Setting the two angles equal to each other we get

$$\frac{\lambda}{D} = \frac{\delta l}{D \cdot f/\#},$$

and

$$f/\# = \frac{\delta l}{\lambda} = \frac{1.5 \times 10^{-5}\,\mathrm{m}}{5 \times 10^{-7}\,\mathrm{m}} = \boxed{30}.$$

Note that the telescope mirror size does not matter!

Problem 4

a) Energy density of the microwave background is $u_R = aT^4$. It can also be written as $u_R = n_\gamma h\nu_{\rm av}$, where n_γ is the photon density, and $\nu_{\rm av}$ is the average frequency of the photons. Approximating

$$\nu_{\rm av} \sim \nu_{\rm max} = 5.88 \times 10^{10} \mathrm{THz/K},$$

we get the following photon density

$$n_\gamma = \frac{aT^4}{h\nu_{av}} \simeq \boxed{382\ \frac{\rm photons}{\rm cm^3}}.$$

b) At the present time the radiation energy density is

$$u_R(\Lambda = 0) = aT^4 = 7.56 \times 10^{-15}\,\frac{\rm ergs}{\rm cm^3 K^4} \times (2.7\,{\rm K})^4 = \boxed{4.02 \times 10^{-13}\,{\rm erg/\,cm^3}}.$$

At a readshift $\Lambda = 1000$, the temperature of the radiation was greater by a factor of Λ, so

$$u_R(\Lambda = 1000) = a(T\Lambda)^4 = \boxed{0.402\ \frac{\rm erg}{\rm cm^3}}.$$

c) Matter energy density today is simply

$$u_M = 2 \times 10^{-31}\frac{g}{\rm cm^3} \times c^2 = 1.8 \times 10^{-10}\frac{\rm erg}{\rm cm^3}.$$

In the past the volume was smaller by a factor of λ^3, but the amount of matter did not change. Then $u_M(\Lambda) = u_M \cdot \Lambda^3$.

As we showed in part (b), radiation density at the present time is $u_R = 4.02 \times 10^{-13}\,{\rm erg/\,cm^3}$ and at redshift Λ it was $u_R(\Lambda) = u_R\Lambda^4$.

To find out when the energy density of matter is equal to that of radiation we write

$$u_R(\Lambda) = u_M(\Lambda);$$

$$u_R\Lambda^4 = u_M\Lambda^3;$$

$$\Lambda = \frac{u_M}{u_R} = \frac{1.8 \times 10^{-10}\,{\rm erg/\,cm^3}}{4.02 \times 10^{-13}\,{\rm erg/\,cm^3}} \simeq \boxed{450}.$$