Astronomy 45. Introduction to Astrophysics

Problem 1

As it turns out, this problem can be done very easily, if you read the notes carefully. On page 73 there are two equations that give L and E in terms of r_0:

$$r_0 = \frac{L^2}{GM\mu^2}, \quad \text{and} \quad \epsilon^2 = 1 + \frac{2EL^2}{(GM)^2\mu^3};$$

and on page 74, geometrical arguments give us the relationship between r_0 and the semimajor axis, $r_0(1-\epsilon^2) = a$. The desired result follows fairly trivially from these three equations.

However, there is a better way to solve this problem practically not relying on the notes, which a few of you have found. The key is to realize that both energy and angular momentum are conserved quantities, so that they have the same value at each point of the orbit. We will consider two special points, perigee and apogee, where $\dot{r}$ is equal to 0, so that the velocity is perpendicular to the radial vector and L is easy to compute

$$L = \mu(\mathbf{v} \times \mathbf{r}) = \mu v r,$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass. At perigee $r_p = a(1-\epsilon)$, and at apogee $r_a = a(1+\epsilon)$. Since the angular momentum is the same at both points, we have the following relation between the velocities

$$\mu v_p r_p = \mu v_a r_a \quad \Longrightarrow \quad \frac{v_p}{v_a} = \frac{1+\epsilon}{1-\epsilon}.$$

In addition, the energy conservation gives us

$$E = \frac{\mu v_p^2}{2} - \frac{GM\mu}{r_p} = \frac{\mu v_a^2}{2} - \frac{GM\mu}{r_a}.$$

Substituting the result from above, we obtain the expression for one of the velocities

$$v_a = \sqrt{\frac{GM(1-\epsilon)}{a(1+\epsilon)}}.$$

From here it is easy to compute both L and E. We have

$$L = \mu v_a r_a = \mu\sqrt{\frac{GM(1-\epsilon)}{a(1+\epsilon)}}a(1+\epsilon) = \mu\sqrt{GMa(1-\epsilon)(1+\epsilon)} =$$

$$= \mu\sqrt{GMa(1-\epsilon^2)} = \mu\sqrt{GMr_0}$$

and

$$E = \frac{\mu v_a^2}{2} - \frac{GM\mu}{r_a} = \frac{\mu}{2}\frac{GM(1-\epsilon)}{a(1+\epsilon)} - \frac{GM\mu}{a(1+\epsilon)} = \frac{GM\mu}{a}\left(\frac{1-\epsilon}{2(1+\epsilon)} - \frac{1}{1+\epsilon}\right) =$$

$$= -\frac{GM\mu}{2a}.$$

Problem 2

a) We know that $L = \mu\sqrt{GMa(1-\epsilon^2)}$. Constant energy orbits means that a is constant. Therefore, to maximize L we need to maximize $(1-\epsilon^2)$, which obviously happens at $\epsilon = 0$. So circular orbit has the most angular momentum for a given energy.

b) Energy of an orbit is simply $-GM\mu/a$. However, we need to take into account the fact that L is kept constant. Using the result of problem 1, $a = L^2/\mu^2 GM(1-\epsilon^2)$, so

$$E = -\frac{(GM)^2\mu^3(1-\epsilon^2)}{L^2}.$$

In this equation, all the quantities except ϵ are constant, so minimum of E (the largest binding energy) clearly occurs at $\epsilon - 0$. So for a given value of L, circular orbit is the most tightly bound one.

c) For simplicity, let's consider circular orbits. The total energy of a spacecraft at distance a from the Earth's center is $E(a) = -GM_\oplus/2a$. At twice that distance, the energy is $E(2a) = -GM_\oplus/4a$, so the spacecraft has to *gain* energy to get to a higher orbit. To do that with a thruster, we need to start by *increasing* our velocity initially, even though at the end, we are going to end up with smaller velocity at a higher orbit.

The most efficient way to get from one orbit to another is as follows (one of you actually found this method!). First, we go from a circular obit at radius a to an elliptical orbit which has its perigee at a, and apogee at $2a$, i.e. it touches both our initial and final orbits. To do that we simply fire a rocket in the direction opposite our direction of motion, thereby increasing our velocity to match that of the elliptical orbit. Note that at perigee, the radial velocity of a particle in an elliptical orbit is zero, so we just have to match the tangential component.

The next step is to go from an elliptical orbit to a circular orbit at radius $2a$. To do this, we wait till we reach apogee (so that we are at $r = 2a$). At this point, our velocity is too low to keep us in a circular orbit, so we again fire a rocket in the direction opposite our direction of motion. Again, we need to worry only about the tangential component, since our radial velocity is instantaneously zero. Having done this, we are now in a desired circular orbit.

Why is this most efficient way of accomplishing the change? It is actually pretty easy to see. The point is that we did not waste any energy. Every time we fired a rocket, we increased our velocity by a maximum amount, since we always fired a thruster directly opposite our direction of motion. To see it more clearly let's actually compute the amount of energy we used.

1. Our original velocity was $v_1 = \sqrt{GM_\oplus/a}$. The elliptical orbit we wanted to enter has a semimajor axis $(a+2a)/2 = 3a/2$ and eccentricity $1 - a/(3a/2) = 1/3$. Then the velocity at perigee is just (see problem 1)

$$v_p = \sqrt{\frac{GM_\oplus(1+\epsilon)}{\frac{3}{2}a(1-\epsilon)}} = \sqrt{\frac{4GM_\oplus}{3a}},$$

so the energy required for acceleration is just

$$E_1 = \frac{mv_p^2}{2} - \frac{mv_1^2}{2} = \frac{GM_\oplus m}{6a},$$

where m is the mass of the spacecraft.

2. Our velocity at apogee in the elliptical orbit is

$$v_a = \sqrt{\frac{GM_\oplus(1-\epsilon)}{\frac{3}{2}a(1+\epsilon)}} = \sqrt{\frac{GM_\oplus}{3a}},$$

and the velocity necessary to stay in the circular orbit is of course $v_2 = \sqrt{GM_\oplus/2a}$. Therefore, the required energy change here is

$$E_2 = \frac{mv_2^2}{2} - \frac{mv_a^2}{2} = \frac{GM_\oplus m}{12a}.$$

So the total energy required is

$$E_1 + E_2 = \frac{GM_\oplus m}{4a},$$

which is just equal to the difference in energy between our initial and final orbits, i.e. the minimum energy required for this change.

d) At perigee, the momentum change $\Delta p = m\Delta v$, corresponds to the following energy change

$$\Delta E_p = \frac{m}{2}(v_p + \Delta v)^2 - \frac{GM_\oplus m}{r_p} - \frac{mv_p^2}{2} + \frac{GM_\oplus m}{r_p} = \frac{m}{2}(2v_p\Delta v + \Delta v^2).$$

Similarly, at apogee, the energy change is

$$\Delta E_a = \frac{m}{2}(2v_a\Delta v + \Delta v^2).$$

In order to escape to infinity, we want to maximize ΔE. The only term that is different in the two cases, is the cross term, so ΔE is maximized when the initial energy is largest. We know that $v_p > v_a$, so clearly one needs to make a momentum change at perigee.

e) For a geosynchronous orbit, $P = 24$ hours. For a circular orbit

$$P = \frac{2\pi a}{v} = \frac{2\pi a^{3/2}}{\sqrt{GM_\oplus}},$$

so

$$a = \left(\frac{GM_\oplus P^2}{4\pi^2}\right)^{1/3} = 4.23 \times 10^9 \text{ cm}.$$

The altitude is $h = a - R_\oplus = 4.23 \times 10^9 \text{ cm} - 6.38 \times 10^8 \text{ cm} = 3.6 \times 10^9 \text{ cm}$.

Since any satellite has to orbit around the Earth's center, geosynchronous orbits are possible only above the equator. Otherwise, it remains fixed in longitude, but not latitude.

Problem 3

a) Using the result from problem 1, the total orbital angular momentum of the system is

$$L = \mu\sqrt{G(M_\odot + M_J)a(1-\epsilon^2)} =$$

$$= \frac{M_\odot\, 0.00095 M_\odot}{1.00095 M_\odot}\sqrt{G 1.00095 M_\odot \times 5.2\,\mathrm{AU}(1 - 0.048^2)} = 1.94 \times 10^{50}\,\mathrm{g\,cm^2\,s^{-1}}.$$

b) Distance from the Sun to the center of mass of the system is

$$a_\odot = a\frac{M_J}{M_\odot + M_J} = 7.4 \times 10^{10}\,\mathrm{cm},$$

just slightly larger than the solar radius.

Assuming that the Sun moves in circular orbit, it's angular momentum is just

$$L_\odot = M_\odot v_\odot a_\odot = \frac{M_\odot 2\pi a_\odot^2}{P} = 1.84 \times 10^{47}\,\mathrm{g\,cm^2\,s^{-1}}.$$

c) Distance from Jupiter to the center of mass of the system is just $a_J = a - a_\odot \simeq a$. So the angular momentum contribution from Jupiter's orbital motion is

$$L_J = M_J v_J a = \frac{M_j 2\pi a^2}{P} = 1.94 \times 10^{50}\,\mathrm{g\,cm^2\,s^{-1}}.$$

d) The moment of inertia of a solid sphere is actually $I = \frac{2}{5}mr^2$, not $I = \frac{2}{5}mr^3$ as given in the problem set. The rotational engular momentum is simply $L = I\omega$, so

$$L_\odot = I_\odot \frac{2\pi}{P_\odot} = \frac{4\pi M_\odot R_\odot^2}{5P_\odot} = 1.1 \times 10^{49}\,\mathrm{g\,cm^2\,s^{-1}},$$

and

$$L_J = I_J \frac{2\pi}{P_J} = \frac{4\pi M_J R_J^2}{5P_J} = 6.7 \times 10^{45}\,\mathrm{g\,cm^2\,s^{-1}}.$$

e) Clearly, the orbital motion of Jupiter dominates the angular momentum of the Sun-Jupiter system. But note that the Sun's rotational angular momentum is only 10 times smaller, so if the Sun were a rapidly rotating star, it would dominate the system.