Astronomy 45. Introduction to Astrophysics

Problem 1

a) We use the equation of motion for θ (page 72 of the notes)

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

to change the variable of integration from t to θ. The new limits of integration are easy to determine – we integrate over the entire orbit, so θ must change from 0 to 2π. Then we get

$$\langle V\rangle = -\frac{1}{\tau}\int_0^\tau \left(\frac{GM\mu}{r}\right) dt = -\frac{1}{\tau}\int_0^{2\pi} \left(\frac{GM\mu}{r}\right) \frac{\mu r^2}{L} d\theta = -\frac{GM\mu^2}{\tau L}\int_0^{2\pi} r d\theta$$

where the expression outside the integral is a constant, as required, and we replaced the forgotten minus sign.

b) To integrate the expression in part (a) we need to write r as a function of θ, using the equation describing the shape of an elliptical orbit (page 73 of the notes):

$$\frac{1}{r} = \frac{1}{r_0}(1 + \epsilon\cos(\theta - \theta_0)).$$

Note that θ_0 is just another constant, so by a change of variables $\theta \to \theta - \theta_0$ we can eliminate it; the limits of integration do not change, since we are still integrating over a complete orbit. After these substitutions, the expression for the average potential energy becomes

$$\langle V\rangle = -\frac{GM\mu^2}{\tau L}\int_0^{2\pi} \frac{r_0 d\theta}{1 + \epsilon\cos(\theta)} = -\frac{GM\mu^2}{\tau L}\frac{2\pi r_0}{\sqrt{1-\epsilon^2}},$$

where we made use of the known definite integral given in the problem set.

Finally, we know that

$$L = \mu\sqrt{GMa(1-\epsilon^2)}, \quad \tau = \frac{2\pi a^{3/2}}{GM}, \text{ and } r_0 = a(1-\epsilon^2),$$

so we can rewrite the expression for $\langle V\rangle$ in a simple form:

$$\langle V\rangle = -\frac{GM\mu^2}{\mu\sqrt{GMa(1-\epsilon^2)}}\frac{2\pi a(1-\epsilon^2)}{\sqrt{1-\epsilon^2}}\frac{\sqrt{GM}}{2\pi a^{3/2}} = -\frac{GM\mu}{a} = 2E,$$

where $E = -GM\mu/2a$ is the total orbital energy.

c) By definition of the total energy, we have $V + K = E$ at all times. Then, simply by taking a time average of both sides, we get $\langle V\rangle + \langle T\rangle = \langle E\rangle = E$, since E is a constant.

In part (b) we proved that $\langle V\rangle = 2E$, then

$$\langle V\rangle + 2\langle T\rangle = \langle V\rangle + 2(E - \langle V\rangle) = 2E - \langle V\rangle = 2E - 2E = 0,$$

as required by the virial theorem.

Problem 2

a) Force on the farther small body is

$$F_1 = \frac{GMm}{R^2},$$

force on the closer one is

$$F_2 = \frac{GMm}{(R+d)^2}.$$

The attractive force between the two small bodies is

$$F_{12} = \frac{Gm^2}{d^2}.$$

We need to determine d such that $F_1 - F_2 = F_{12}$.

$$GMm\left[\frac{1}{R^2} - \frac{1}{(R+d)^2}\right] = \frac{Gm^2}{d^2};$$

$$\frac{M}{m}\left[\frac{R^2 + d^2 + 2dR - R^2}{R^2(R+d)^2}\right] = \frac{1}{d^2};$$

$$\frac{M}{m}\left[\frac{(d+2R)d}{(R+d)^2}\right] = \frac{R^2}{d^2}.$$

To proceed, we need to make some approximations. We know that $M \gg m$, let's assume that $R \gg d$. Then $d + 2R \simeq 2R$ and $(R+d)^2 \simeq R^2$. We get

$$\frac{M}{m}\frac{2Rd}{R^2} \simeq \frac{R^2}{d^2},$$

$$\frac{2M}{m} \simeq \frac{R^3}{d^3};$$

and finally

$$d \simeq R\left[\frac{m}{2M}\right]^{1/3}.$$

Note that as long as $M \gg m$, this result is valid.

b) The result here is going to be to some degree a function of our initial assumptions. For simplicity, let's consider a body of mass m and radius r as the sum of its two halfs. Then this problem reduces to the one we solved in part (a), except that now we have two bodies each of mass $m/2$ separated by a distance $\sim r$. Then, we can use the relation derived above:

$$r \simeq R \left[\frac{m}{4M}\right]^{1/3}$$

or

$$R \simeq r \left[\frac{4M}{m}\right]^{1/3},$$

where R is the desired distance between the two bodies. Note that one can get a slightly different numerical factor due to a different interpretation of the problem, but the functional dependence should be the same.

c) The limiting factor here is the solar radius, i.e. we want planets to be disrupted before they enter the Sun's atmosphere, where they will be evaporated before tides need to be taken into account. Thus we need

$$R \simeq r \left[\frac{4M_\odot}{m}\right]^{1/3} > R_\odot.$$

Since $M_\odot$ and $R_\odot$ are fixed, the important quantity is the fraction $r/m^{1/3}$ which is proportional to $\rho^{-1/3}$, where ρ is the planet's density. Then the above relationship is just a condition on ρ

$$r^3 \frac{4M_\odot}{m} = \frac{4\pi r^3}{3m}\frac{3M_\odot}{\pi} = \frac{3M_\odot}{\rho\pi} > R_\odot^3;$$

$$\rho < \frac{3M_\odot}{\pi R_\odot^3} = 5.6\,\mathrm{g/cm^3}.$$

It turns out that all planets in the solar system fall into two categories: all terrestrial planets (except Pluto) have densities $\sim 5\,\mathrm{g/cm^3}$, and all gas giants and Pluto are much less dense with $\rho \sim 1\,\mathrm{g/cm^3}$. Thus, according to our calculation, any of the gas giants will be torn apart before entering the solar atmosphere, whereas it is a borderline case for terrestrial planets.

What about a solar mass white dwarf star? Then we have a different condition on planetary densities:

$$\rho < \frac{3M_\odot}{\pi (R_\odot/100)^3} = 25.6\,\mathrm{g/cm^3}.$$

This calculation shows that any solar system planet can certainly be disrupted well outside the surface of a white dwarf star.

d) The condition we use here is that a black hole must remain *outside* the star, i.e. $R > d$, where R is the distance from the hole to the center of the star, and $d = R_\odot$ is the stellar radius; we want the star to be tidally disrupted, not completely sucked into the hole. However, we cannot use the approximate expression we derived in part (b), since we do not know whether the condition on the masses is satisfied in this case. Instead we go one step back and start with

$$\frac{2M_{BH}}{M_\odot}\left[\frac{(R_\odot + 2R)R_\odot}{(R+R_\odot)^2}\right] = \frac{R^2}{R_\odot^2},$$

where again we think of a star as a sum of its two halfs. To find the smallest possible value for M_{BH} we set $R \simeq R_\odot$:

$$M_{BH} \simeq M_\odot \frac{R^2(R+R_\odot)^2}{R_\odot^3(R_\odot + 2R)} > M_\odot \frac{4R_\odot^4}{3R_\odot^4} = \frac{4}{3}M_\odot.$$

Thus to disrupt any star, a black hole has to have a minimum mass a few times greater than that star. This is of course a very intuitive result, we could have guessed the answer without any calculations.

What about the maximum mass? Now we have another condition, R has to be greater than the black hole horizon radius R_h. In this case we can go back to our assumption that $M_{BH} \gg M_\odot$. Then we can write

$$R \simeq R_\odot\left[\frac{4M_{BH}}{M_\odot}\right]^{1/3} > \frac{2GM_{BH}}{c^2};$$

Rewriting this as a condition on M_{BH} we get

$$M_{BH} < \sqrt{\frac{4c^6R_\odot^3}{8G^3M_\odot}} \simeq 2\times 10^8 M_\odot.$$

Black holes with mass bigger than this limit still disrupt stars, but that happens inside the black hole horizon and therefore is completely unobservable. Thus, larger black holes are in some sense worse at disrupting stars.

Why does this happen? Well, according to the expression we derived in part (b), tidal radius grows as $M_{BH}^{1/3}$, and the radius of the horizon is linear in M_{BH} so it grows much faster with increasing black hole mass and eventually the tidal radius becomes hidden inside the horizon.

Problem 3

a) Suppose a parcel gas with mass m starts out at the L_1 Lagrangian point distance a from the secondary with no initial velocity. It's initial potential energy with respect to the secondary is approximately $V_i \simeq -GM_2m/a$. It's final potential energy is $V_f = -GM_2m/R_2$. Then since the total energy is conserved we must have $V_i = V_f + T_f$, where T_f is the kinetic energy of the material at the surface of the secondary and

$$T_f = GMm\left(\frac{1}{R_2} - \frac{1}{a}\right) = \alpha\frac{GMm}{R_2},$$

where we defined $\alpha = (a - R_2)/a$.

The rate of kinetic energy release is just

$$L = \frac{dT_f}{dt} = \alpha\frac{GM\dot{m}}{R_2},$$

where $\dot{m} = dm/dt$. If the matter is assumed to fall from infinity, i.e. $a \gg R_2$, then $\alpha = 1$ as required. However, in a real physical system that might not be the case. For example if we are considering a "close" binary system, where a is roughly comparable to R_2, α will clearly be much less than 1. Similarly, if the matter is accreting via an accretion disk instead of just falling directly onto the secondary, some of the potential energy will be dissipated in the disk through friction and radiated away instead of going into heating of the secondary. This effect tends to produce $\alpha \sim 0.5$.

b) If the emerging radiation is a blackbody, we must have

$$L = 4\pi R_2^2\sigma T_2^4.$$

Then

$$T_2 = \left[\frac{L}{4\pi R_2^2\sigma}\right]^{1/4} = \left[\frac{\alpha GM\dot{m}}{4\pi R_2^3\sigma}\right]^{1/4}.$$

c) First let's convert $\dot{m}$ into cgs units.

$$\dot{m} = 10^{-8}M_\odot/\mathrm{yr} = \frac{10^{-8}\ M_\odot/\mathrm{yr} \times 2\times 10^{33}\ \mathrm{g}/M_\odot}{3.16\times 10^7\ \mathrm{s/yr}} = 6.3\times 10^{17}\ \mathrm{g/s}.$$

The accretion luminosity is then

$$L = \frac{6.67\times 10^{-8}\ \mathrm{dyne\ cm^2}/g^2 \times 2\times 10^{33}\mathrm{g} \times 6.3\times 10^{17}\ \mathrm{g/s}}{7\times 10^{10}\ \mathrm{cm}}\ \mathrm{ergs/s} =$$

$$= 1.2\times 10^{33}\ \mathrm{ergs/s} = 0.3L_\odot.$$

Therefore, the accretion luminosity is comparable to the solar luminosity, so that it is not a good assumption to ignore the internal energy generation of such a star.

The characteristic frequency corresponds to the peak of the blackbody curve.

$$T_2 = \left[\frac{1.2 \times 10^{33}\,\text{ergs/s}}{4\pi(7 \times 10^{10}\ \text{cm})^2 \times 5.67 \times 10^{-5}\ \text{erg/ cm}^2\text{/ s/ K}^4} \right]^{1/4} = 4300\,\text{K}$$

Then

$$\nu = 5.88 \times 10^{10}\,\text{Hz/K}\ T_2 = 2.5 \times 10^{14}\,\text{Hz}.$$

d)

White Dwarf

For a white dwarf star the accretion luminosity becomes

$$L_{WD} = \frac{GM_\odot \dot{m}}{R_{WD}} = 8.4 \times 10^{34}\,\text{ergs/s} \simeq 21 L_\odot$$

The effective temperature of the secondary is now

$$T_{WD} \simeq 10^5\,\text{K},$$

so that the characteristic frequency is

$$\nu_{WD} \simeq 5.88 \times 10^{15}\,\text{Hz}.$$

These systems are called cataclysmic variables (CVs) and are best observed in UV.

Neutron Star

$$L_{NS} = \frac{GM_\odot\, 1.4\dot{m}}{R_{NS}} = 1.2 \times 10^{38}\,\text{ergs/s} \simeq 29,400 L_\odot$$

The effective temperature of the secondary is now

$$T_{NS} \simeq 2 \times 10^7\,\text{K},$$

so that the characteristic frequency is

$$\nu_{NS} \simeq 10^{18}\,\text{Hz}.$$

Thus, these systems are very bright and are best observed in X-rays. Astronomers call them X-ray binaries for obvious reason.