Astronomy 45. Introduction to Astrophysics

Problem 1

a) For $n = 0$ Lane-Emden equation takes the simple form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -1.$$

After multiplying both sides by $\xi^2\, d\xi$, we can integrate it directly:

$$\int d\left(\xi^2\frac{d\phi}{d\xi}\right) = -\int \xi^2\, d\xi,$$

$$\xi^2\frac{d\phi}{d\xi} = -\frac{\xi^3}{3} + c_1,$$

where c_1 is a constant of integration.

Now we have a first order differential equation. To integrate it, we multiply both sides by $d\xi/\xi^2$.

$$\int d\phi = \int\left(-\frac{\xi}{3} + \frac{c1}{\xi^2}\right) d\xi,$$

$$\phi = -\frac{\xi^2}{6} - \frac{c1}{\xi} + c_2,$$

where c_2 is another constant of integration.

To determine the values for c_1 and c_2, we have to remember our boundary conditions. They are

$$\frac{d\phi}{d\xi} \qquad \text{and} \qquad \phi(0) = 1\bigg|_{\xi=0} = 0.$$

The first boundary condition is a bit tricky. The derivative of ϕ with respect to ξ is of course

$$\frac{d\phi}{d\xi} = -\frac{\xi}{3} + \frac{c_1}{\xi^2},$$

this was in fact one of our intermediate steps. Now if we set $\xi = 0$, the second term on the right hand side diverges. The only way to keep it from exploding is by setting $c_1 = 0$. Applying the second boundary condition gives us $c_2 = 1$.

Thus our final result is

$$\phi = -\frac{\xi^2}{6} + 1.$$

As a check, we can compute ξ_1 and $\xi_1^2\phi_1'$:

$$\xi_1 = \xi(\phi = 0) = \sqrt{6} \simeq 2.449$$

and

$$\xi_1^2\phi_1' = \xi_1^2 \frac{d\phi}{d\xi}\bigg|_{\xi=\xi_1} = -\frac{\xi_1^3}{3} = -2\sqrt{6} \simeq 4.899.$$

Both values are in agreement with the table on page 131.

b) Let's start with the density as a function of radius. By definition of ϕ, $\rho(r) = \rho_c\phi^n = \rho_c$. So the density is constant throughout the star and is equal to the average value

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}.$$

There are two ways to compute $P(r)$. We can use the polytropic equation of state, even though it seems to diverge at a first glance,

$$P(r) = K\rho(r)^{1+\frac{1}{n}} = K\rho_c^{1+\frac{1}{n}}\phi^{n+1} = \left(K\rho_c^{1+\frac{1}{n}}\right)\phi.$$

We know that P must be finite at any point in the star, thus the quantity in parentheses is just a constant, which is equal to $P_c = P(0)$ (since $\phi(0) = 1$). The expression for P_c is derived in the notes (page 132):

$$P_c = \frac{1}{4\pi}\frac{1}{(\phi_1')^2}\frac{GM^2}{R^4} = \frac{1}{4\pi}\left(\frac{3}{\sqrt{6}}\right)^2\frac{GM^2}{R^4} = \frac{3}{8\pi}\frac{GM^2}{R^4},$$

where we substituted the numerical value for $(\phi_1')^2$ computed in part (a). So

$$P(r) = \frac{3}{8\pi}\frac{GM^2}{R^4}\phi = \frac{3}{8\pi}\frac{GM^2}{R^4}\left[1 - \frac{1}{6}\left(\frac{r}{a}\right)^2\right].$$

The expression for a in terms of K and ρ_c (page 128) also seems to diverge for $n = 0$. But we can easily eliminate this divergence:

$$a^2 = \frac{K\rho_c^{\frac{1}{n}-1}}{4\pi G} = \frac{K\rho_c^{\frac{1}{n}+1}}{4\pi G\rho_c^2} = \frac{3}{8\pi}\frac{GM^2}{R^4}\frac{\left(\frac{4}{3}\pi R^3\right)^2}{4\pi GM^2} = \frac{R^2}{6}.$$

Finally, when we combine the results of the last two equations, we get

$$P(r) = \frac{3}{8\pi}\frac{GM^2}{R^4}(1 - \frac{r^2}{R^2}) = \frac{3}{8\pi}\frac{GM^2}{R^6}(R^2 - r^2).$$

Another and somewhat easier way to get the same result is by integrating the equation for hydrostatic equilibrium. This is easily done for $n = 0$ model since the density is constant throughout the star. Most of you used this method. The main thing to remember here is that we need to use $M(r)$ and *not* the total mass of the star:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} = -\frac{4\pi r^3 G\rho_c^2}{3r^2} = -\frac{3GM^2}{4\pi R^6}r.$$

Now we multiply both sides by dr and integrate (don't forget the integration constant which is equal to the central pressure):

$$P = -\frac{3GM^2}{8\pi R^6}r^2 + P_c = \frac{3GM^2}{8\pi R^6}(R^2 - r^2).$$

.

Finally, the temperature can be determined from the perfect gas law equation

$$T(r) = \frac{\mu m_p}{k}\frac{P(r)}{\rho(r)} = \frac{\mu m_p}{k}\frac{3GM^2}{8\pi R^6}\frac{\frac{4}{3}\pi R^3}{M}(R^2 - r^2) = \frac{GM}{2R^3}(R^2 - r^2),$$

So $n = 0$ polytrope model is just a uniform density sphere.

Problem 2

a) First of all, let's figure out what is the difference between ${}^{12}_{6}$C and ${}^{56}_{26}$Fe. The only relevant quantity is the mean molecular weight per electron, μ_e, which is exactly 2 for C and $56/26 \simeq 2.15$ for Fe, since iron has slightly more neutrons than protons.

Now consider a non-relativistic white dwarf star. First thing we need to compare is $\rho(r)$ for two stars of *the same* ρ_c (this point is very important, in fact most of you were confused exactly what quantity to hold constant). Since $\phi(\xi)$ is fixed for a given polytropic index and ρ_c is constant, $\rho(\xi)$ will be the same for both Fe and C white dwarfs. But not $\rho(r)$! Because $r = \xi a$, where $a \propto \sqrt{K} \propto \mu_e^{-5/6}$ for fixed ρ_c (see pages 128 and 141). Thus, $a_{\rm Fe} < a_{\rm C}$, and for a given ξ, $r_{\rm Fe} < r_{\rm C}$. Since density decreases monotonically with radius, this implies that $\rho_{\rm Fe}(r) < \rho_{\rm C}(r)$ at a fixed $r > 0$. Physically this just means that in the iron white dwarf density falls off faster with radius than in the carbon white dwarf. An immediate consequence of this result is that Fe white dwarf will have a smaller radius.

To compare $M(r)$ in the two cases remember that $M(r) = \int_0^r 4\pi r^2 \rho(r) dr$. Since $\rho_{\rm Fe}(r) < \rho_{\rm C}(r)$ for all $r > 0$, this clearly implies that $M_{\rm Fe}(r) < M_{\rm C}(r)$.

To compare pressure is a little bit more tricky. In the center, since ρ_c is the same in both cases, $P_{\rm Fe}$ is clearly less than $P_{\rm C}$ (just because we have less electrons in the center of the iron star). At $r > 0$ we have

$$P(r) \propto \left(\frac{\rho(r)}{\mu_e}\right)^{5/3},$$

and since $\rho_{\rm Fe}(r) < \rho_{\rm C}(r)$ and $\mu_e({\rm Fe}) > \mu_e({\rm C})$, we must have $P_{\rm Fe}(r) < P_{\rm C}(r)$. Alternatively, we can argue that since pressure has to support the weight of the outer layers of the star, and for a given r there is less material on top in the iron white dwarf than in the carbon one, pressure in the former is less.

b) Clearly, since iron is able to provide less electron degeneracy pressure per unit mass, $M_{\rm Ch}$ must be smaller. To compute the fractional difference remember that $M_{\rm Ch} \propto \mu_e^{-2}$. Then

$$\frac{M_{\rm Ch,C}}{M_{\rm Ch,Fe}} = \left(\frac{\mu_e({\rm Fe})}{\mu_e({\rm C})}\right)^2 = \left(\frac{2.15}{2}\right)^2 \simeq 1.16.$$

Problem 3

a) Number of neutrinos created during the inverse β-decay is

$$N_{\nu_e} = \frac{1.4M_\odot}{m_p + m_e} \simeq \frac{1.4M_\odot}{m_p} = 1.8 \times 10^{57}.$$

Each one carries an energy of $\sim 1\,{\rm MeV} = 10^6\,{\rm eV} \times 1.6 \times 10^{-12}\,{\rm ergs/eV} = 1.6 \times 10^{-6}{\rm ergs}$, so the total energy loss is

$$E_{\nu_e} = 1.8 \times 10^{57} \times 1.6 \times 10^{-6}{\rm ergs} = 2.9 \times 10^{51}{\rm ergs}.$$

b) The initial radius of the white dwarf (assuming non-relativistic) equation of state can be easily computed using the equation on page 142:

$$R_{\rm WD} = 10^9\,{\rm cm}\left[\frac{0.7011{\rm M}_\odot}{1.4{\rm M}_\odot}\left(\frac{2}{\mu_{\rm e}}\right)^5\right]^{1/3} = 7.9 \times 10^8\,{\rm cm}.$$

Note that this is rather inexact, since we know that $1.4M_\odot$ white dwarfs *are* relativistic. However, as we will see later, the exact number doesn't really matter here.

To compute the radius of the neutron star we need to go back to the equation of state on page 141. It is derived for electron degeneracy pressure, but the result for neutron degeneracy pressure must be practically the same, except we have to substitute $\mu_n \to \mu_e$ and $m_n \to m_e$. Then

$$P_n = \left[\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_n m_p^{5/3}\mu_n^{5/3}}\right]\rho^{5/3} \simeq \left[\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_p^{8/3}}\right]\rho^{5/3},$$

where we set $m_n \simeq m_p$, and of course mean molecular weight per neutron, μ_n, is unity.

Now we follow the same sequence of steps as we did for a non-relativistic white dwarf (page 142) with $n = 3/2$. Since $P_n = K_n\rho^{1+\frac{1}{n}}$,

$$K_n = \left[\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_p^{8/3}}\right].$$

And using the expressions for R and M derived from polytrope theory,

$$R_{\rm NS} = \xi_1 \left[\frac{5K_n}{8\pi G}\right]^{1/2} \rho_c^{-1/6} = 1.45 \times 10^6 \,{\rm cm} \left(\frac{10^{15}\,{\rm g/cm^3}}{\rho_{\rm c}}\right)^{1/6}$$

and

$$M_{\rm NS} = -\xi_1^2 \phi_1' \, 4\pi \left[\frac{5K_n}{8\pi G}\right]^{3/2} \rho^{1/2} = 1.11 M_\odot \left[\frac{\rho_c}{10^{15}\,{\rm g/cm^3}}\right]^{1/2}.$$

After eliminating ρ_c, we have an expression for R in terms of M

$$R_{\rm NS} = 1.5 \times 10^6 \left(\frac{M_\odot}{M_{\rm NS}}\right)^{1/3}.$$

For $M_{\rm NS} = 1.4 M_\odot$, $R_{\rm NS} = 13.4$ km.

The gravitational binding energy for a star of mass M and radius R is roughly $\sim GM^2/R$, so energy released in the collapse is

$$E_g \sim -\frac{GM_{\rm WD}^2}{R_{\rm WD}} + \frac{GM_{\rm NS}^2}{R_{\rm NS}} = G(1.4M_\odot)^2 \left(\frac{1}{R_{\rm NS}} - \frac{1}{R_{\rm WD}}\right) \simeq \frac{G(1.4M_\odot)^2}{R_{\rm NS}},$$

since $R_{\rm NS} \ll R_{\rm WD}$. So

$$E_g \sim 3.9 \times 10^{53}\,{\rm ergs}.$$

Thus, the energy lost through neutrinos is on the order of 1% of the total energy released in the collapse.

c) The mass of the envelope is $M_{env} = 8.6 M_\odot$. Since $E_{\nu_e} \ll E_g$,

$$\frac{M_{env} v^2}{2} = E_g$$

and the velocity of the ejected envelope is

$$v = \sqrt{\frac{2E_g}{M_{env}}} = 6.7 \times 10^9\,{\rm cm/s} = 0.22{\rm c},$$

fairly relativistic.

d) Both protons and electrons have the same velocity because the plasma must be neutral. Since protons have larger mass, they have larger kinetic energy and correspondingly larger temperature.

$$T_p \simeq \frac{m_p v^2}{2k} = 2.7 \times 10^{11}\ {\rm K},$$

and

$$T_e \simeq \frac{m_e v^2}{2k} = 1.5 \times 10^8\ {\rm K}.$$

Radiation is emitted by the electrons, so T_e is the relevant temperature. Thermal radiation is always emitted at a characteristic blackbody frequency

$$\nu_{max} \sim 5.9 \times 10^{10} T_e \,\mathrm{Hz} = 8.8 \times 10^{18} \,\mathrm{Hz}.$$

Thus, observations in X-rays are most appropriate for studying these shock fronts.